

A PROOF OF ANDREWS' q -DYSON CONJECTURE FOR $n = 4$

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ABSTRACT. Andrews' q -Dyson conjecture is that the constant term in a polynomial associated with the root system A_{n-1} is equal to the q -multinomial coefficient. Good used an identity to establish the case $q = 1$, which was originally raised by Dyson. Andrews established his conjecture for $n \leq 3$ and Macdonald proved it when $a_1 = a_2 = \cdots = a_n = 1, 2$ or ∞ for all $n \geq 2$. We use a q -analog of Good's identity which involves a remainder term and linear algebra to establish the conjecture for $n = 4$. The remainder term arises because of an essential problem with the q -Dyson conjecture: the symmetry of the constant term. We give a number of conjectures related to the symmetry.

1. Introduction and summary. Let $n \geq 2$ and $a_i \geq 0$, $1 \leq i \leq n$, be nonnegative integers and set

$$(1.1) \quad g_n(\mathbf{a}; \mathbf{x}) = \prod_{1 \leq i < j \leq n} \left(q \frac{x_j}{x_i} \right)_{a_j} \left(\frac{x_i}{x_j} \right)_{a_i},$$

where q is fixed with $|q| < 1$ and

$$(1.2) \quad \begin{aligned} (x)_0 &= 1, \\ (x)_m &= \prod_{i=0}^{m-1} (1 - xq^i), \quad m \geq 1, \\ (x)_\infty &= \lim_{m \rightarrow \infty} (x)_m = \prod_{i=0}^{\infty} (1 - xq^i). \end{aligned}$$

Andrews [1] conjectured that

$$(1.3) \quad \text{C.T. } g_n(\mathbf{a}, \mathbf{x}) = \frac{(q)_{(a_1 + a_2 + \cdots + a_n)}}{(q)_{a_1} (q)_{a_2} \cdots (q)_{a_n}},$$

where C.T. g is the constant term in the Laurent expansion of g in powers of x_1, x_2, \dots, x_n . The case $q = 1$ of (1.3) was originally raised by Dyson [13] and settled independently by Gunson [18] and Wilson [39]. It is

$$(1.4) \quad \text{C.T. } f_n(\mathbf{a}, \mathbf{x}) = \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!},$$

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where

$$(1.5) \quad f_n(\mathbf{a}, \mathbf{x}) = \prod_{1 \leq i < j \leq n} \left(1 - \frac{x_j}{x_i}\right)^{a_j} \left(1 - \frac{x_i}{x_j}\right)^{a_i}.$$

Good [17] gave an elegant induction proof of (1.4) using the identity

$$(1.6) \quad 1 = \sum_{k=1}^n \prod_{\substack{i=1 \\ i \neq k}}^n \frac{1}{(1 - x_k/x_i)}.$$

Good [16] also used MacMahon's master theorem [34, Article 66] to establish (1.4) for $n = 3$. This gives an alternative proof of Dixon's formula [12] (see Bailey [6, §3.1(1)]). Andrews [1] established (1.3) for $n = 3$, thus giving another derivation of a q -analog of Dixon's formula due to Jackson [22] (see Carlitz [9]). We shall establish the q -Dyson conjecture (1.3) for $n = 4$ by using a q -analog of Good's identity (1.6) which has a remainder term.

Macdonald [32, 33] proved that (1.3) holds when $a_1 = a_2 = \dots = a_n = 1, 2$ or ∞ for all $n \geq 2$. He observed that (1.3) is related to the root system A_{n-1} and established corresponding results for other root systems. He established [33] the case $q = 1$ of his conjectures for the root systems A_{n-1} , B_n , C_n , D_n and BC_n by using Selberg's [38] integral

$$(1.7) \quad \int_0^1 \dots \int_0^1 \prod_{i=1}^n t_i^{(x-1)} (1 - t_i)^{(y-1)} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2z} dt_1 \dots dt_n \\ = \prod_{i=1}^n \frac{\Gamma(x + (i-1)z) \Gamma(y + (i-1)z) \Gamma(1 + iz)}{\Gamma(x + y + (n+i-2)z) \Gamma(1 + z)},$$

where $\operatorname{Re}(x) > 0$, $\operatorname{Re}(y) > 0$ and $\operatorname{Re}(z) > -[1/n, \operatorname{Re}(x)/(n-1), \operatorname{Re}(y)/(n-1)]$. Askey [4] gave a number of conjectured q -analogs of Selberg's integral (1.7) which imply (1.3) when $a_1 = a_2 = \dots = a_n$. Andrews [3] gave an integral formulation of (1.4). Askey [4] gave a conjectured q -analog of Andrews' integral which implies (1.3). See Morris [36] for other constant term conjectures and Evans [14] for character sum analogs of many of these results. Askey [5] proved Morris' conjecture for BC_1 .

In order to treat the remainder term, we must understand that it arises because of a basic problem with the q -Dyson conjecture: symmetry. Since the q -multinomial coefficient

$$(1.8) \quad C_n(\mathbf{a}) = \frac{(q)_{(a_1+a_2+\dots+a_n)}}{(q)_{a_1}(q)_{a_2}\dots(q)_{a_n}}$$

on the right side of (1.3) is symmetric in a_1, a_2, \dots, a_n , we must have

$$(1.9) \quad G_n(\pi(\mathbf{a})) = G_n(\mathbf{a}), \quad \pi(\mathbf{a}) = (a_{\pi(1)}, a_{\pi(2)}, \dots, a_{\pi(n)}),$$

where $\pi \in S_n$ and

$$(1.10) \quad G_n(\mathbf{a}) = \text{C.T. } g_n(\mathbf{a}; \mathbf{x}).$$

When $q = 1$, the symmetry of (1.4) follows easily from

$$(1.11) \quad f_n(\pi(\mathbf{a}); \pi(\mathbf{x})) = f_n(\mathbf{a}; \mathbf{x}).$$

However, the zeros of

$$(1.12) \quad g_n(\pi(\mathbf{a}); \pi(\mathbf{x})) = \prod_{\substack{1 \leq i < j \leq n \\ \pi(i) < \pi(j)}} \left(q \frac{x_j}{x_i} \right)_{a_j} \left(\frac{x_i}{x_j} \right)_{a_i} \prod_{\substack{1 \leq i < j \leq n \\ \pi(i) > \pi(j)}} \left(\frac{x_j}{x_i} \right)_{a_j} \left(q \frac{x_i}{x_j} \right)_{a_i}$$

are obtained by a small but intricate shifting of those of $g_n(\mathbf{a}; \mathbf{x})$. The only simple q -analogs of (1.11) are

$$(1.13) \quad g_n(a_n, a_1, \dots, a_{n-1}; qx_n, x_1, \dots, x_{n-1}) = g_n(\mathbf{a}; \mathbf{x})$$

and its iterates. Clearly $G_n(\mathbf{a})$ is fixed

$$(1.14) \quad G_n(\tau(\mathbf{a})) = G_n(\mathbf{a}), \quad \tau(\mathbf{a}) = (a_n, a_1, \dots, a_{n-1}),$$

by the n -cycle τ occurring in (1.13) and its powers.

In §2, we obtain Good's identity (1.6) directly from the Vandermonde determinant, thus relating (1.6) to the root system A_{n-1} . We use an asymmetric q -Vandermonde which arises from the factors of $g_n(\mathbf{a}; \mathbf{x})$ whose zeros are shifted in (1.12) for some $\pi \in S_n$. The polynomial $\Sigma_n^*(\mathbf{a}; \mathbf{x})$ arises from and has a structure which reflects the asymmetry of our q -Vandermonde. Andrews [3] and Zeilberger [40] have commented on the difficulties of finding a q -analog of (1.6). We give a q -analog of (1.6) which has a remainder term involving $\Sigma_n^*(\mathbf{a}; \mathbf{x})$. We conjecture that the contribution to $G_n(\mathbf{a})$ arising from each term of $\Sigma_n^*(\mathbf{a}; \mathbf{x})$ is 0. We give some related conjectures which have varying amounts of symmetry.

In §3, we use brute force to show that certain weighted averages of the remainder term in our q -analog of Good's identity (1.6) are 0 for $n \leq 5$. Kadell [25] has shown that the first part of Conjecture 5 holds for all $n \geq 2$. This shows that the q -Dyson conjecture (1.3) is equivalent (see [25]) to the symmetry (1.9). We show that (1.3) holds if $a_1 = a_2 = \dots = a_{m-1} = a_{m+1} = \dots = a_n = 1$ or ∞ , where $1 \leq m \leq n$. This works because (1.9) reduces to (1.14) at every stage of our induction. We establish (1.3) for $n = 3$ since the required symmetry (1.9) follows easily using the well-known q -binomial theorem.

In §4, we see that our simple argument which establishes the symmetry (1.9) for $n = 3$ does not work for $n = 4$. By (1.14), $G_4(\pi(\mathbf{a}))$ can only assume six possible values where $\pi \in S_4$. If there is some equality among a_i , $1 \leq i \leq 4$, then there are fewer possibilities. We use the relations obtained in §3 to show that the possible values of $G_4(\pi(\mathbf{a}))$, $\pi \in S_4$, satisfy the required number of linearly independent equations. This establishes (1.3) for $n = 4$.

In §5, we discuss some possible approaches to treating the case $n \geq 5$ and their ramifications in light of the symmetry (1.9).

2. A q -analog of Good's identity. We have

$$(2.1) \quad \begin{aligned} V_n(\mathbf{x}) &= \prod_{1 \leq i < j \leq n} (x_i - x_j) = \det |x_j^{(n-i)}|_{n \times n} \\ &= \sum_{\pi \in S_n} \text{sgn}(\pi) \prod_{i=1}^n x_i^{(n-\pi(i))}. \end{aligned}$$

Expanding the Vandermonde determinant along the bottom row yields

$$(2.2) \quad \prod_{1 \leq i < j \leq n} (x_i - x_j) = \sum_{k=1}^n (-1)^{(n-k)} \prod_{\substack{i=1 \\ i \neq k}}^n x_i \prod_{\substack{1 \leq i < j \leq n \\ i \neq k \neq j}} (x_i - x_j)$$

and dividing both sides of (2.2) by $V_n(\mathbf{x})$ gives Good's identity (1.6). Multiplying (1.6) by $f_n(\mathbf{a}; \mathbf{x})$ yields

$$(2.3) \quad f_n(\mathbf{a}; \mathbf{x}) = \sum_{k=1}^n f_n(a_1, a_2, \dots, a_{k-1}, a_k - 1, a_{k+1}, \dots, a_n; \mathbf{x}) \\ = \sum_{k=1}^n f_n(\mathbf{a} - \mathbf{e}_k; \mathbf{x}),$$

where \mathbf{e}_k is the standard unit vector whose k th coordinate is 1. To complete Good's proof of (1.4) [17], we need only equate constant terms in (2.3) and verify a simple boundary condition.

Good's identity (1.6) is a direct consequence of the expansion (2.2) of the Vandermonde $V_n(\mathbf{x})$. We commence our attack on the q -Dyson conjecture (1.3) by expanding the q -Vandermonde

$$(2.4) \quad V_n(\mathbf{a}; \mathbf{x}) = \prod_{1 \leq i < j \leq n} (x_i - q^{a_j} x_j).$$

Observe that the expression in the sum on the right side of (2.2) contains those terms in the expansion of $V_n(\mathbf{x})$ which do not contain x_k as a factor. Let $\Sigma_n^*(\mathbf{a}; \mathbf{x})$ be the sum of all of the terms in the expansion of $V_n(\mathbf{a}; \mathbf{x})$ in which each of the variables x_1, x_2, \dots, x_n occurs. Any other term must have exactly one variable missing since either x_i or x_j is introduced by the factor $(x_i - q^{a_j} x_j)$. We obtain

$$(2.5) \quad \prod_{1 \leq i < j \leq n} (x_i - q^{a_j} x_j) \\ = \sum_{k=1}^n (-1)^{(n-k)} q^{[\sum_{k < j \leq n} a_j]} \prod_{\substack{i=1 \\ i \neq k}}^n x_i \prod_{\substack{1 \leq i < j \leq n \\ i \neq k \neq j}} (x_i - q^{a_j} x_j) + \Sigma_n^*(\mathbf{a}; \mathbf{x}).$$

To extend (2.1), let $\Sigma_n^{**}(\mathbf{a}; \mathbf{x})$ be the sum of all of the terms in the expansion of $V_n(\mathbf{a}; \mathbf{x})$ in which two of the variables x_1, x_2, \dots, x_n occur to the same power. This includes all of the terms in $\Sigma_n^*(\mathbf{a}; \mathbf{x})$. We obtain

$$(2.6) \quad \prod_{1 \leq i < j \leq n} (x_i - q^{a_j} x_j) = \sum_{\pi \in S_n} \prod_{\substack{1 \leq i < j \leq n \\ \pi(i) < \pi(j)}} x_i \prod_{\substack{1 \leq i < j \leq n \\ \pi(i) > \pi(j)}} (-q^{a_j} x_j) + \Sigma_n^{**}(\mathbf{a}; \mathbf{x}).$$

It will be convenient to multiply (2.5) and (2.6) by $\prod_{1 \leq i < j \leq n} x_i^{(-1)}$. This yields

$$(2.7) \quad \prod_{1 \leq i < j \leq n} \left(1 - q^{a_j} \frac{x_j}{x_i}\right) \\ = \sum_{k=1}^n q^{[\sum_{k < j \leq n} a_j]} \prod_{k < j \leq n} \left(-\frac{x_j}{x_k}\right) \prod_{\substack{1 \leq i < j \leq n \\ i \neq k \neq j}} \left(1 - q^{a_j} \frac{x_j}{x_i}\right) \\ + \prod_{1 \leq i < j \leq n} x_i^{(-1)} \Sigma_n^*(\mathbf{a}; \mathbf{x})$$

$$(2.8) \quad = \sum_{\pi \in S_n} \prod_{\substack{1 \leq i < j \leq n \\ \pi(i) > \pi(j)}} \left(-q^{a_j} \frac{x_j}{x_i} \right) + \prod_{1 \leq i < j \leq n} x_i^{(-1)} \sum_n^{**}(\mathbf{a}; \mathbf{x}).$$

Set

$$(2.9) \quad \hat{g}_n(\mathbf{a}; \mathbf{x}) = \prod_{1 \leq i < j \leq n} \left(q \frac{x_j}{x_i} \right)_{a_j-1} \left(\frac{x_i}{x_j} \right)_{a_i}$$

and observe that

$$(2.10) \quad g_n(\mathbf{a}; \mathbf{x}) = \hat{g}_n(\mathbf{a}; \mathbf{x}) \prod_{1 \leq i < j \leq n} \left(1 - q^{a_j} \frac{x_j}{x_i} \right).$$

We want to substitute (2.7) into (2.10). We have

$$(2.11) \quad \begin{aligned} & \prod_{k < j \leq n} \left(-\frac{x_j}{x_k} \right) \prod_{\substack{1 \leq i < j \leq n \\ i \neq k \neq j}} \left(1 - q^{a_j} \frac{x_j}{x_i} \right) \hat{g}_n(\mathbf{a}; \mathbf{x}) \\ &= \prod_{1 \leq i < k} \left(q \frac{x_k}{x_i} \right)_{a_k-1} \left(\frac{x_i}{x_k} \right)_{a_i} \prod_{k < j \leq n} \left(-\frac{x_j}{x_k} \right) \left(q \frac{x_j}{x_k} \right)_{a_j-1} \left(\frac{x_k}{x_j} \right)_{a_k} \\ & \quad \cdot \prod_{\substack{1 \leq i < j \leq n \\ i \neq k \neq j}} \left(1 - q^{a_j} \frac{x_j}{x_i} \right) \left(q \frac{x_j}{x_i} \right)_{a_j-1} \left(\frac{x_i}{x_j} \right)_{a_i} \\ &= \prod_{1 \leq i < k} \left(q \frac{x_k}{x_i} \right)_{a_k-1} \left(\frac{x_i}{x_k} \right)_{a_i} \prod_{k < j \leq n} \left(\frac{x_j}{x_k} \right)_{a_j} \left(q \frac{x_k}{x_j} \right)_{a_k-1} \\ & \quad \cdot \prod_{\substack{1 \leq i < j \leq n \\ i \neq k \neq j}} \left(q \frac{x_j}{x_i} \right)_{a_j} \left(\frac{x_i}{x_j} \right)_{a_i} \\ &= g_n(a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n, a_k - 1; x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n, x_k) \\ &= g_n(\pi_k(\mathbf{a} - \mathbf{e}_k); \pi_k(\mathbf{x})), \end{aligned}$$

where

$$(2.12) \quad \pi_k(\mathbf{x}) = (x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_n, x_k).$$

Let

$$(2.13) \quad \begin{aligned} H_n(\mathbf{a}; \mathbf{x}) &= \prod_{1 \leq i < j \leq n} x_i^{(-1)} \hat{g}_n(\mathbf{a}; \mathbf{x}) \\ &= \prod_{1 \leq i < j \leq n} \left(\frac{1}{x_i} - \frac{1}{x_j} \right) \left(q \frac{x_j}{x_i} \right)_{a_j-1} \left(q \frac{x_i}{x_j} \right)_{a_i-1}. \end{aligned}$$

We now use (2.11) and (2.13) to substitute (2.7) into (2.10). This yields

$$(2.14) \quad g_n(\mathbf{a}; \mathbf{x}) = \sum_{k=1}^n q^{[\sum_{k < j \leq n} a_j]} g_n(\pi_k(\mathbf{a} - \mathbf{e}_k); \pi_k(\mathbf{x})) + R_n(\mathbf{a}; \mathbf{x}),$$

where the remainder $R_n(\mathbf{a}; \mathbf{x})$ is given by

$$(2.15) \quad R_n(\mathbf{a}; \mathbf{x}) = \sum_n^* (\mathbf{a}; \mathbf{x}) H_n(\mathbf{a}; \mathbf{x}).$$

Observe that, for all $\pi \in S_n$,

$$(2.16) \quad \hat{g}_n(\pi(\mathbf{a}); \pi(\mathbf{x})) = \prod_{\substack{1 \leq i < j \leq n \\ \pi(i) > \pi(j)}} \left(-\frac{x_j}{x_i} \right) \hat{g}_n(\mathbf{a}; \mathbf{x}).$$

A similar argument using (2.8) and (2.16) yields

$$(2.17) \quad g_n(\mathbf{a}; \mathbf{x}) = \sum_{\pi \in S_n} q^{[\sum_{1 \leq i < j \leq n, \pi(i) > \pi(j)} a_j]} \hat{g}_n(\pi(\mathbf{a}); \pi(\mathbf{x})) + \sum_n^{**}(\mathbf{a}; \mathbf{x}) H_n(\mathbf{a}; \mathbf{x}).$$

If $a_k = 0$ where $1 \leq k \leq n$, then x_k can only occur in the numerator of a term in the Laurent expansion of $g_n(\mathbf{a}; \mathbf{x})$. Thus, we have the boundary conditions

$$(2.18) \quad \begin{aligned} G_n(a_1, a_2, \dots, a_{k-1}, 0, a_{k+1}, \dots, a_n) \\ = G_{n-1}(a_1, a_2, \dots, a_{k-1}, a_{k+1}, \dots, a_n), \\ G_1(0) = 1. \end{aligned}$$

$C_n(\mathbf{a})$ is uniquely determined by (2.18) and the q -difference equation

$$(2.19) \quad \begin{aligned} C_n(\mathbf{a}) &= \sum_{k=1}^n q^{[\sum_{k < j \leq n} a_j]} C_n(\mathbf{a} - \mathbf{e}_k) \\ &= \sum_{k=1}^n q^{[\sum_{k < j \leq n} a_j]} C_n(\pi_k(\mathbf{a} - \mathbf{e}_k)). \end{aligned}$$

The first equality in (2.19) is well-known. The second uses the symmetry of $C_n(\mathbf{a})$. We would be able to use our q -analog (2.14) of Good's identity (1.6) to prove the q -Dyson conjecture (1.3) if only we knew that the constant term in the Laurent expansion of the remainder $R_n(\mathbf{a}; \mathbf{x})$ were 0. Impressive computer evidence for $n = 4$ suggests

Conjecture 1. If $\omega_n(\mathbf{a}; \mathbf{x})$ is a term of $\sum_n^{**}(\mathbf{a}; \mathbf{x})$, then

$$(2.20) \quad \text{C.T. } \omega_n(\mathbf{a}; \mathbf{x}) H_n(\mathbf{a}; \mathbf{x}) = 0.$$

The q -Dyson conjecture (1.3) follows from the weaker version of Conjecture 1 which holds for $\sum_n^*(\mathbf{a}; \mathbf{x})$.

The original motivation for the choice (2.4) of $V_n(\mathbf{a}; \mathbf{x})$ was the fact that (2.5) and (2.19) both involve the same power of q . $\hat{g}_n(\mathbf{a}; \mathbf{x})$ in (2.9) was obtained by deleting the factors of $g_n(\mathbf{a}; \mathbf{x})$ which have a zero in common with $V_n(\mathbf{a}; \mathbf{x})$. The real motivation should have been (1.12). $V_n(\mathbf{a}; \mathbf{x})$ vanishes at all of the zeros of $g_n(\mathbf{a}; \mathbf{x})$ for which there is some $\pi \in S_n$ such that $g_n(\pi(\mathbf{a}); \pi(\mathbf{x}))$ does not vanish. $\hat{g}_n(\mathbf{a}; \mathbf{x})$ vanishes precisely when $g_n(\pi(\mathbf{a}); \pi(\mathbf{x}))$ vanishes for all $\pi \in S_n$.

We have

Conjecture 2.

$$(2.21) \quad \text{C.T. } \hat{g}_n(\mathbf{a}; \mathbf{x}) = \prod_{i=2}^n \frac{(1 - q^{a_i})}{(1 - q^{a_1 + a_2 + \dots + a_i})} C_n(\mathbf{a}).$$

Observe that the q -Dyson conjecture (1.3) is recovered by equating constant terms in (2.17) and using Conjectures 1 and 2.

Despite the symmetry (2.16) of its zeros, C.T. $\hat{g}_n(\mathbf{a}; \mathbf{x})$ is entirely asymmetric. To obtain varying amounts of symmetry, let $1 \leq m \leq n$ and set

$$(2.22) \quad g_{n,m}(\mathbf{a}; \mathbf{x}) = \prod_{1 \leq i < j \leq n} \left(q \frac{x_j}{x_i} \right)_{a_j - \chi(j \leq m)} \left(\frac{x_i}{x_j} \right)_{a_i},$$

where $\chi(A)$ is 1 or 0 according to whether A is true or false. We have

Conjecture 3.

$$(2.23) \quad \text{C.T. } g_{n,m}(\mathbf{a}; \mathbf{x}) = \prod_{i=2}^m \frac{(1 - q^{a_i})}{(1 - q^{a_1 + a_2 + \dots + a_i})} C_n(\mathbf{a}),$$

which is symmetric in $a_{m+\chi(m \neq 1)}, \dots, a_n$. The case $m = 1$ of Conjecture 3 is the q -Dyson conjecture (1.3) and the case $m = n$ is Conjecture 2. Good's identity (1.6) can be used to prove the case $q = 1$ of Conjectures 1, 2 and 3. See Barton and Mallows [7] to avoid a tedious induction. Kadell [26] gives some related conjectured extensions of Selberg's integral (1.7). Kadell [24] discusses the q -difference equation (2.19) and expresses the right side of (2.23) as a generating function. The symmetry of $C_n(\mathbf{a})$ and the basic n -cycle τ occur [24, (3.11)] naturally.

3. Symmetry. Observe that for $1 \leq k \leq n$, q^{a_k} occurs in $k - 1$ of the factors of our q -Vandermonde $V_n(\mathbf{a}; \mathbf{x})$. From (2.5) and (2.6), we see that

$$(3.1) \quad \# \left(\sum_n^* (\mathbf{a}; \mathbf{x}) \right) = 2^{(n(n-1)/2)} - n 2^{((n-1)(n-2)/2)}$$

and

$$(3.2) \quad \# \left(\sum_n^{**} (\mathbf{a}; \mathbf{x}) \right) = 2^{(n(n-1)/2)} - n!,$$

where $\#(P)$ is the number of terms in the polynomial P . Thus $\sum_n^*(\mathbf{a}; \mathbf{x})$ and the larger polynomial $\sum_n^{**}(\mathbf{a}; \mathbf{x})$ contain most of the terms of $V_n(\mathbf{a}; \mathbf{x})$ and they inherit an apparent asymmetry. Set $q^{a_i} = A_i$ and let $a_1, a_2, \dots, q^{a_1} = A_1, q^{a_2} = A_2, \dots$ and x_1, x_2, \dots be represented by a, b, \dots, A, B, \dots and S, T, \dots , respectively. Then

$$(3.3) \quad \sum_2^*(a, b; S, T) = \sum_2^{**}(a, b; S, T) = 0,$$

$$(3.4) \quad \sum_3^*(a, b, c; S, T, U) = \sum_3^{**}(a, b, c; S, T, U) = (BC - C)STU,$$

$$(3.5)$$

$$\begin{aligned} \sum_4^*(a, b, c, d; S, T, U, V) = & (CD - D)S^3TUV + (BD - BCD)ST^3UV \\ & + (BC^2D - C^2D)STU^3V + (CD^3 - BCD^3)STUV^3 \\ & + (BD - D + CD - BCD)S^2T^2UV \\ & + (2CD - BCD - C^2D)S^2TU^2V \\ & + (D^2 - 2CD^2 + BCD^2)S^2TUV^2 \\ & + (CD - 2BCD + BC^2D)ST^2U^2V \\ & + (2BCD^2 - BD^2 - CD^2)ST^2UV^2 \\ & + (BCD^2 - CD^2 + C^2D^2 - BC^2D^2)STU^2V^2 \end{aligned}$$

and
(3.6)

$$\begin{aligned} \sum_4^{**}(a, b, c, d; S, T, U, V) &= \sum_4^*(a, b, c, d; S, T, U, V) \\ &\quad + (BC - C)S^2T^2U^2 + (D^2 - BD^2)S^2T^2V^2 \\ &\quad + (C^2D^2 - CD^2)S^2U^2V^2 + (BCD^2 - BC^2D^2)T^2U^2V^2. \end{aligned}$$

It is clear from (2.13) that, for every $\pi \in S_n$,

$$(3.7) \quad H_n(\pi(\mathbf{a}); \pi(\mathbf{x})) = \text{sgn}(\pi) H_n(\mathbf{a}; \mathbf{x}).$$

We define the antisymmetrization Ξ_n of $P_n(\mathbf{a}; \mathbf{x})$ by

$$(3.8) \quad \Xi_n(P_n(\mathbf{a}; \mathbf{x})) = \sum_{\pi \in S_n} \text{sgn}(\pi) P_n(\pi(\mathbf{a}); \pi(\mathbf{x})).$$

Observe that

$$(3.9) \quad \sum_{\pi \in S_n} W_n(\pi(\mathbf{a})) R_n(\pi(\mathbf{a}); \pi(\mathbf{x})) = \Xi_n\left(W_n(\mathbf{a}) \sum_n^*(\mathbf{a}; \mathbf{x})\right) H_n(\mathbf{a}; \mathbf{x}).$$

We want to find functions $W_n(\mathbf{a})$ for which the weighted average (3.9) of the remainder term is 0. We have

LEMMA 4. *Let $P_n(\mathbf{a}; \mathbf{x}) = \sum_{\alpha \in \Omega} \omega_n^\alpha(\mathbf{a}; \mathbf{x})$, where each $\omega_n^\alpha(\mathbf{a}; \mathbf{x})$ is a monomial in $-A_1, -A_2, \dots, -A_n$ and x_1, x_2, \dots, x_n and all of the coefficients are equal. (Observe that $V_n(\mathbf{a}; \mathbf{x})$ and hence all of the polynomials we will deal with have this property.) Then*

$$(3.10) \quad \Xi_n(P_n(\mathbf{a}; \mathbf{x})) = 0$$

if and only if there exists an involution Ψ of Ω and a mapping $\alpha \rightarrow \pi_\alpha$ of Ω into S_n such that each π_α is odd and

$$(3.11) \quad \omega_n^\alpha(\mathbf{a}; \mathbf{x}) = \omega_n^{\Psi(\alpha)}(\pi_\alpha(\mathbf{a}); \pi_\alpha(\mathbf{x})).$$

PROOF. If (3.11) holds, then (3.10) follows easily using

$$(3.12) \quad \Xi_n(\omega_n(\mathbf{a}; \mathbf{x})) = \text{sgn}(\pi) \Xi_n(\omega_n(\pi(\mathbf{a}); \pi(\mathbf{x}))),$$

since each π_α is odd. If (3.10) holds, then we may construct the required Ψ and π_α as follows. Choose $\alpha \in \Omega$ and let $\omega_n^\alpha(\mathbf{a}; \mathbf{x})$ be cancelled by the term for α' and π

$$\omega_n^\alpha(\mathbf{a}; \mathbf{x}) + \text{sgn}(\pi) \omega_n^{\alpha'}(\pi(\mathbf{a}); \pi(\mathbf{x})) = 0$$

in the expansion (3.8) of $\Xi_n(P_n(\mathbf{a}; \mathbf{x}))$. π must be odd by our hypothesis that all of the coefficients are equal. Set $\Psi(\alpha) = \alpha'$, $\pi_\alpha = \pi$ and remove α from Ω . If $\alpha' \neq \alpha$, set $\Psi(\alpha') = \alpha$, $\pi_{\alpha'} = \pi^{(-1)}$ and remove α' from Ω . We now have a smaller polynomial which antisymmetrizes to 0. We may repeat as required to obtain Ψ and π_α . \square

The following results have been obtained by the computer using Lemma 4:

$$(3.13a) \quad 0 = \Xi_3\left(\sum_3^*(\mathbf{a}; \mathbf{x})\right)$$

$$(3.13b) \quad = \Xi_3\left(A \sum_3^*(\mathbf{a}; \mathbf{x})\right)$$

$$(3.13c) \quad = \Xi_3\left(AB \sum_3^*(\mathbf{a}; \mathbf{x})\right)$$

$$(3.13d) \quad = \Xi_3\left(A^2B \sum_3^*(\mathbf{a}; \mathbf{x})\right),$$

$$(3.14a) \quad 0 = \Xi_4 \left(\sum_4^* (\mathbf{a}; \mathbf{x}) \right)$$

$$(3.14b) \quad = \Xi_4 \left(AB \sum_4^* (\mathbf{a}; \mathbf{x}) \right)$$

$$(3.14c) \quad = \Xi_4 \left(A^2 BC \sum_4^* (\mathbf{a}; \mathbf{x}) \right)$$

$$(3.14d) \quad = \Xi_4 \left(A^3 B^2 C \sum_4^* (\mathbf{a}; \mathbf{x}) \right),$$

$$(3.15a) \quad 0 = \Xi_4 \left((A^2 BCD + ABC) \sum_4^* (\mathbf{a}; \mathbf{x}) \right)$$

$$(3.15b) \quad = \Xi_4 \left((A^3 B^2 CD + A^2 B^2 C) \sum_4^* (\mathbf{a}; \mathbf{x}) \right),$$

$$(3.16a)$$

$$0 = \Xi_4 \left((A^3 BCD - A^2 B^3 CD + A^2 BC^2 D - A^3 BC + AB^2 C) \sum_4^* (\mathbf{a}; \mathbf{x}) \right)$$

$$(3.16b)$$

$$= \Xi_4 \left((A^3 B^2 CD^2 - A^3 B^2 C^2 + A^2 BC^2 D - A^2 B^2 D + A^2 B^2 C^2) \sum_4^* (\mathbf{a}; \mathbf{x}) \right),$$

$$(3.17a) \quad 0 = \Xi_5 \left(\sum_5^* (\mathbf{a}; \mathbf{x}) \right)$$

$$(3.17b) \quad = \Xi_5 \left(A^4 B^3 C^2 D \sum_5^* (\mathbf{a}; \mathbf{x}) \right).$$

The results (3.13a)–(3.13d) for $n = 3$ are easily established directly from definition (3.8) or by Lemma 4. Ψ is always the identity map. For $n \geq 4$, this appears to happen only in the context of Lemma 7. We may establish (3.14a)–(3.14d) and (3.17a)–(3.17b) by taking all of the π_α to be transpositions. We cannot establish (3.15a)–(3.15b) or (3.16a)–(3.16b), which are essential in our proof of the case $n = 4$, without taking some of the π_α to be 4-cycles. All of the results (3.13a)–(3.17b) (and our comments) hold with $\sum_n^*(\mathbf{a}; \mathbf{x})$ replaced by $\sum_n^{**}(\mathbf{a}; \mathbf{x})$.

(3.13a), (3.13d), (3.14a), (3.14d), (3.17a) and (3.17b) suggest

Conjecture 5.

$$(3.18a) \quad 0 = \Xi_n \left(\sum_n^* (\mathbf{a}; \mathbf{x}) \right)$$

$$(3.18b) \quad = \Xi_n \left(\prod_{i=1}^{n-1} A_i^{(n-i)} \sum_n^* (\mathbf{a}; \mathbf{x}) \right).$$

The stronger version of Conjecture 5 which holds for $\sum_n^{**}(\mathbf{a}; \mathbf{x})$ extends the result of Carter [10, Theorem 10.2.1] due to Macdonald [31] for the root system A_{n-1} . Kadell [25] establishes (3.18a) for all $n \geq 2$. This shows that the q -Dyson conjecture (1.3) is equivalent to the symmetry (1.9) of the constant term $G_n(\mathbf{a})$. To see this, we must examine Good's induction [17]. In the q -case, it is natural to try to prove that

$$(3.19) \quad G_n(\pi(\mathbf{a})) = C_n(\mathbf{a})$$

holds for all $\pi \in S_n$. This gives a stronger induction hypothesis which we adopt for the rest of the paper. We proceed by induction on n and $a_1 + a_2 + \cdots + a_n$. By (2.18), we may assume that $a_i \geq 1$, $1 \leq i \leq n$, and that (1.3) holds if we subtract 1

from one of the parameters which are then permuted. Let $1 \leq k \leq n$ and $\pi \in S_n$. We have

$$(3.20) \quad G_n(\pi_k(\pi(\mathbf{a}) - \mathbf{e}_k)) = C_n(\pi_k(\pi(\mathbf{a}) - \mathbf{e}_k)).$$

Replace \mathbf{a} and \mathbf{x} by $\pi(\mathbf{a})$ and $\pi(\mathbf{x})$ in our q -analog (2.14) of Good's identity (1.6). This gives

$$(3.21) \quad \begin{aligned} & g_n(\pi(\mathbf{a}); \pi(\mathbf{x})) \\ &= \sum_{k=1}^n q^{[\sum_{k < j \leq n} a_{\pi(j)}]} g_n(\pi_k(\pi(\mathbf{a}) - \mathbf{e}_k); \pi_k(\pi(\mathbf{x}))) + R_n(\pi(\mathbf{a}); \pi(\mathbf{x})). \end{aligned}$$

Equating constant terms in (3.21) yields

$$(3.22) \quad G_n(\pi(\mathbf{a})) = \sum_{k=1}^n q^{[\sum_{k < j \leq n} a_{\pi(j)}]} G_n(\pi_k(\pi(\mathbf{a}) - \mathbf{e}_k)) + \text{C.T. } R_n(\pi(\mathbf{a}); \pi(\mathbf{x})).$$

Replace \mathbf{a} by $\pi(\mathbf{a})$ in (2.19) and use the symmetry of $C_n(\mathbf{a})$. We obtain

$$(3.23) \quad C_n(\mathbf{a}) = \sum_{k=1}^n q^{[\sum_{k < j \leq n} a_{\pi(j)}]} C_n(\pi_k(\pi(\mathbf{a}) - \mathbf{e}_k)).$$

Using (3.23) and our induction hypothesis (3.20), (3.22) then becomes

$$(3.24) \quad G_n(\pi(\mathbf{a})) = C_n(\mathbf{a}) + \text{C.T. } R_n(\pi(\mathbf{a}); \pi(\mathbf{x})).$$

Multiply (3.24) by $W_n(\pi(\mathbf{a}))$ and use (3.9) to sum over all $\pi \in S_n$. This yields

$$(3.25) \quad \begin{aligned} & \sum_{\pi \in S_n} W_n(\pi(\mathbf{a})) G_n(\pi(\mathbf{a})) \\ &= \left(\sum_{\pi \in S_n} W_n(\pi(\mathbf{a})) \right) C_n(\mathbf{a}) + \text{C.T. } \Xi_n(W_n(\mathbf{a}) \sum_n^*(\mathbf{a}; \mathbf{x})) H_n(\mathbf{a}; \mathbf{x}). \end{aligned}$$

Suppose that $W_n(\mathbf{a})$ satisfies

$$(3.26) \quad \Xi_n(W_n(\mathbf{a}) \sum_n^*(\mathbf{a}; \mathbf{x})) = 0.$$

Then (3.25) becomes

$$(3.27) \quad \sum_{\pi \in S_n} W_n(\pi(\mathbf{a})) G_n(\pi(\mathbf{a})) = \left(\sum_{\pi \in S_n} W_n(\pi(\mathbf{a})) \right) C_n(\mathbf{a}).$$

If (2.20) of Conjecture 1 holds with \mathbf{a} and \mathbf{x} replaced by $\pi(\mathbf{a})$ and $\pi(\mathbf{x})$, then we are done by (3.22). If (3.26) and the symmetry (1.9) hold, then we are done by (3.27). In either case, we have only completed one step of our induction proof. We shall see how far we can carry the induction using simple arguments. We have

LEMMA 6. *Let $1 \leq k \leq n$ and $\omega_n(\mathbf{a}; \mathbf{x})$ be a term of $\sum_n^{**}(\mathbf{a}; \mathbf{x})$. Then there exist i and j with $1 \leq i < j \leq n$, $i \neq k \neq j$, such that x_i and x_j occur to the same power in $\omega_n(\mathbf{a}; \mathbf{x})$.*

PROOF. Assume this fails. The variables x_i , $1 \leq i \leq n$, $i \neq k$, must occur to $n - 1$ distinct powers satisfying $0 \leq e \leq n - 1$. The remaining value must be the exponent of x_k , since the sum of the exponents of all of the variables is $n(n - 1)/2$. Thus, no two variables occur to the same power, which is a contradiction. \square

This gives the simple

LEMMA 7. Let $1 \leq k \leq n$ and $a_1 = a_2 = \cdots = a_{k-1} = a_{k+1} = \cdots = a_n$. Then

- (i) (2.20) of Conjecture 1 holds,
- (ii) (3.18a) of Conjecture 5 holds,
- (iii) $G_n(\mathbf{a})$ is symmetric (1.9).

PROOF. Let i and j satisfy the requirements of Lemma 6.

(i) The function $\omega_n(\mathbf{a}; \mathbf{x})H_n(\mathbf{a}; \mathbf{x})$ changes sign when x_i and x_j are interchanged. Hence, it must have constant term 0.

(ii) Let Ψ be the identity map and π_α the transposition which interchanges i and j . The result follows by Lemma 4.

(iii) Clearly, there exists m , $1 \leq m \leq n$, such that $\pi(\mathbf{a}) = \tau^m(\mathbf{a})$. The result follows by (1.14). \square

We have

COROLLARY 8. Let $1 \leq m \leq n$. The q -Dyson conjecture (1.3) holds if $a_1 = a_2 = \cdots = a_{m-1} = a_{m+1} = \cdots = a_n = 1$ or ∞ .

PROOF. We proceed by induction on n and a_m using our strong induction hypothesis (3.19). By (2.18), we have $a_m \geq 1$. Let the common value be 1. (3.20) holds by our induction assumption on a_m or n according to whether $\pi(k) = m$ and $a_m > 1$ or not. Our induction is completed using Lemma 7(i) and (3.24). For the ∞ case, we may assume by (1.14) that $m = n$. (2.14) becomes

$$(3.28) \quad g_n(\infty, \dots, \infty, a_n; \mathbf{x}) = g_n(\infty, \dots, \infty, a_n - 1; \mathbf{x}) + q^{a_n} g_n(\infty, \dots, a_n, \infty; \pi_{n-1}(\mathbf{x})) + R_n(\infty, \dots, \infty, a_n; \mathbf{x}).$$

We equate constant terms and use (1.14) and Lemma 7(i). This yields

$$(3.29) \quad (1 - q^{a_n})G_n(\infty, \dots, \infty, a_n) = G_n(\infty, \dots, \infty, a_n - 1)$$

and the result follows easily. \square

Corollary 8 also follows from Lemma 7(ii) and (iii). Set $W_n(\mathbf{a}) = 1$. Lemma 7(ii) gives (3.26) and the result (3.19) follows by applying Lemma 7(iii) to (3.27). The reader should check the modifications required for the ∞ case. Since (3.18a) of Conjecture 5 holds [25] for all $n \geq 2$, the q -Dyson conjecture (1.3) follows as long as the symmetry (1.9) holds for all of the preceding stages of our induction. We treat the easy case $n = 3$.

We have the well-known (see Andrews [2, (2.2.1)]) q -binomial theorem

$$(3.30) \quad (qx)_m = \sum_{k=0}^m (-x)^k q^{k(k+1)/2} \frac{(q)_m}{(q)_k (q)_{m-k}}.$$

This gives us the Laurent expansion of a polynomial with only simple zeros which form a geometric sequence. Observe that x^a times

$$(3.31) \quad (qx)_b \left(\frac{1}{x} \right)_a = \sum_{k=-a}^b (-x)^k q^{k(k+1)/2} \frac{(q)_{a+b}}{(q)_{a+k} (q)_{b-k}}$$

is such a polynomial, so that (3.31) is equivalent to (3.30). This gives (1.3) for $n = 2$. If we let a and b tend to ∞ in (3.31) and multiply by $(q)_\infty$, we obtain the Jacobi triple product formula [23]

$$(3.32) \quad (q)_\infty (qx)_\infty \left(\frac{1}{x}\right)_\infty = \sum_{k=-\infty}^{\infty} (-x)^k q^{k(k+1)/2}.$$

This proof goes back to Cauchy [11, pp. 50–55] and Gauss [15, pp. 461–469]. Andrews [1] used (3.31) to expand $g_3(\mathbf{a}; \mathbf{x})$ and extract

$$(3.33) \quad G_3(a, b, c) = \sum_{k=-M}^{M^*} (-1)^k q^{k(3k+1)/2} \frac{(q)_{a+b} (q)_{a+c} (q)_{b+c}}{(q)_{a-k} (q)_{b-k} (q)_{c-k} (q)_{a+k} (q)_{b+k} (q)_{c+k}},$$

where $M = \min(a, b, c)$. Since each term on the right side of (3.33) is symmetric in a, b, c , we have proved

THEOREM 9 (ANDREWS [1]). *The q -Dyson conjecture (1.3) holds for $n = 3$.*

4. $n = 4$. We can expand each factor of $g_n(\mathbf{a}; \mathbf{x})$ by (3.31) and extract $G_n(\mathbf{a})$ as a multiple sum. For the case $q = 1$ see Andrews [3] and Dyson [13], who uses this representation to establish his original conjecture (1.4) for $n \leq 5$. Unfortunately, for $n \geq 4$ this expansion of $G_n(\mathbf{a})$ fails in a spectacular way to exhibit any symmetry beyond (1.14). Let $P_n(\mathbf{a})$ be the sum of all of the terms contributing to $G_n(\mathbf{a})$ which have a positive coefficient. Thus, $P_3(a, b, c)$ is obtained by taking the sum of the terms on the right side of (3.33) for k even. Let $N_n(\mathbf{a})$ be given similarly so that

$$(4.1) \quad G_n(\mathbf{a}) = P_n(\mathbf{a}) - N_n(\mathbf{a}).$$

The first case in which the symmetry (1.9) is open to question arises for $\mathbf{a} = (1, 2, 1, 2)$ and $\mathbf{a} = (1, 1, 2, 2)$. The computer gives

$$(4.2) \quad \begin{aligned} P_4(1, 2, 1, 2) &= 1 + 7q + 29q^2 + 80q^3 + 162q^4 \\ &\quad + 255q^5 + 318q^6 + 318q^7 + 255q^8 \\ &\quad + 162q^9 + 80q^{10} + 29q^{11} + 7q^{12} + q^{13} \\ &= P_4(1, 1, 2, 2) - q^4 + q^6 + q^7 - q^9. \end{aligned}$$

Of course, $N_4(1, 2, 1, 2) = N_4(1, 1, 2, 2) - q^4 + q^6 + q^7 - q^9$ and (4.1) gives the conjectured value (1.3) in both cases. This and a great deal of computer evidence for $n = 4$ suggest

Conjecture 10. Let $n \geq 4$ and $a_i \geq 1$, $1 \leq i \leq n$. Then

$$(4.3) \quad P_n(\pi(\mathbf{a})) = P_n(\mathbf{a})$$

if and only if $\pi(\mathbf{a}) = \tau^k(\mathbf{a})$ for some k , $1 \leq k \leq n$.

A similar result should also hold for $N_n(\mathbf{a})$.

The “if” part of Conjecture 10 is clear from (1.13). Using (3.33), we see that $P_3(a, b, c)$ is symmetric in a, b, c . Thus, the “only if” part of Conjecture 10 does not hold for $n = 3$. For $n \geq 4$, (1.14) represents a real limit on the symmetry (1.9) that

can be established easily, at least using (3.31). We use our relations (3.14a)–(3.14d), (3.15b) and (3.16a) to establish

THEOREM 11. *The q -Dyson conjecture (1.3) holds for $n = 4$.*

PROOF. Let $n = 4$. By (2.18) and Theorem 9, we may assume that $a_i \geq 1$, $1 \leq i \leq 4$. We use our strong induction argument developed in §3. We must show that

$$(4.4) \quad G_4(\pi(\mathbf{a})) = C_4(\mathbf{a})$$

holds for every $\pi \in S_4$. We proceed by induction on $a + b + c + d$. Set

$$(4.5) \quad \begin{aligned} W_4^1(\mathbf{a}) &= 1, & W_4^2(\mathbf{a}) &= AB, & W_4^3(\mathbf{a}) &= A^2BC, \\ W_4^4(\mathbf{a}) &= A^3B^2C, & W_4^5(\mathbf{a}) &= A^3B^2CD + A^2B^2C, \\ W_4^6(\mathbf{a}) &= A^3BCD - A^2B^3CD + A^2BC^2D - A^3BC + AB^2C. \end{aligned}$$

(3.26) follows by our relations (3.14a)–(3.14d), (3.15b) and (3.16a). Our induction again gives (3.27) which is

$$(4.6) \quad \sum_{\pi \in S_4} W_4^s(\pi(\mathbf{a})) G_4(\pi(\mathbf{a})) = \left(\sum_{\pi \in S_4} W_4^s(\pi(\mathbf{a})) \right) C_4(\mathbf{a}), \quad 1 \leq s \leq 6.$$

S_3 is embedded in S_4 by

$$(4.7) \quad \begin{aligned} \sigma_1(1, 2, 3, 4) &= (1, 2, 3, 4), & \sigma_2(1, 2, 3, 4) &= (1, 3, 2, 4), \\ \sigma_3(1, 2, 3, 4) &= (2, 3, 1, 4), & \sigma_4(1, 2, 3, 4) &= (2, 1, 3, 4), \\ \sigma_5(1, 2, 3, 4) &= (3, 1, 2, 4), & \sigma_6(1, 2, 3, 4) &= (3, 2, 1, 4) \end{aligned}$$

and $\{\tau^k | 1 \leq k \leq 4\}$ is a set of coset representatives (on either side). By (1.14), (4.4) reduces to

$$(4.8) \quad G_4(\sigma_t(\mathbf{a})) = C_4(\mathbf{a}), \quad 1 \leq t \leq 6,$$

and (4.6) becomes

$$(4.9) \quad \sum_{t=1}^6 \left(\sum_{k=1}^4 W_4^s(\tau^k(\sigma_t(\mathbf{a}))) \right) G_4(\sigma_t(\mathbf{a})) = \left(\sum_{\pi \in S_4} W_4^s(\pi(\mathbf{a})) \right) C_4(\mathbf{a}),$$

$$1 \leq s \leq 6.$$

If a_i , $1 \leq i \leq 4$, are distinct, then we may establish (4.8) by showing that the six equations (4.9) satisfied by $G_4(\sigma_t(\mathbf{a}))$, $1 \leq t \leq 6$, are linearly independent. When there is some equality among a_i , $1 \leq i \leq 4$, (1.14) gives some equality among $G_4(\sigma_t(\mathbf{a}))$, $1 \leq t \leq 6$, and we require fewer equations. We treat these cases separately. We start with

Case (i). a, b, c, d distinct.

Without loss of generality, we take $1 \leq a < b < c < d$. The 6×6 coefficient matrix C of the system (4.9) is given by

$$(4.10) \quad C = (c_{s,t})_{\substack{1 \leq s \leq 6 \\ 1 \leq t \leq 6}}, \quad c_{s,t} = \sum_{k=1}^4 W_4^s(\tau^k(\sigma_t(\mathbf{a}))).$$

Clearly, $c_{1,t} = 4$ for $1 \leq t \leq 6$. We compute

$$(4.11) \quad \begin{aligned} c_{5,4} &= A^2B^3CD + A^2B^2C + AB^2CD^3 + AB^2D^2 \\ &\quad + ABC^3D^2 + BC^2D^2 + A^3BC^2D + A^2C^2D \\ &= A^2B^2C + \text{higher order terms.} \end{aligned}$$

Thus, $A^2B^2C = q^{(2a+2b+c)}$ is the term with the smallest power of q occurring in $c_{5,4}$. Similarly,

$$(4.12) \quad c_{5,1} = A^2B^2C + \text{higher order terms}$$

and $c_{5,2}$, $c_{5,3}$, $c_{5,5}$ and $c_{5,6}$ contain only higher order terms. It is easy to check that for $s = 2, 3, 4$ or 6 , the term with the smallest possible power of q is AB , A^2BC , A^3B^2C or A^2B^2C , respectively. Such terms, each with coefficient 1, occur only in $c_{2,1}$, $c_{2,4}$, $c_{2,5}$, $c_{2,6}$, $c_{3,1}$, $c_{3,2}$, $c_{4,1}$, $c_{6,4}$ and $c_{6,5}$. Using \cdots to indicate higher order terms, the coefficient matrix C is given below.

$$(4.13)$$

$$C = \begin{vmatrix} \overset{4}{(AB + \cdots)} & \overset{4}{(\cdots)} & \overset{4}{(\cdots)} & \overset{4}{(AB + \cdots)} & \overset{4}{(AB + \cdots)} & \overset{4}{(AB + \cdots)} \\ \overset{4}{(A^2BC + \cdots)} & \overset{4}{(A^2BC + \cdots)} & \overset{4}{(\cdots)} & \overset{4}{(\cdots)} & \overset{4}{(\cdots)} & \overset{4}{(\cdots)} \\ \overset{4}{(A^3B^2C + \cdots)} & \overset{4}{(\cdots)} & \overset{4}{(\cdots)} & \overset{4}{(\cdots)} & \overset{4}{(\cdots)} & \overset{4}{(\cdots)} \\ \overset{4}{(A^2B^2C + \cdots)} & \overset{4}{(\cdots)} & \overset{4}{(\cdots)} & \overset{4}{(A^2B^2C + \cdots)} & \overset{4}{(\cdots)} & \overset{4}{(\cdots)} \\ \overset{4}{(\cdots)} & \overset{4}{(\cdots)} & \overset{4}{(\cdots)} & \overset{4}{(A^2BC + \cdots)} & \overset{4}{(A^2BC + \cdots)} & \overset{4}{(\cdots)} \end{vmatrix}$$

Hence

$$(4.14) \quad \det C = -4A^{10}B^7C^4 + \text{higher order terms} \neq 0,$$

since q is transcendental. Since both sides of (4.8) satisfy the system (4.9), they are equal as required.

Case (ii). a, b, c distinct, $c = d$.

We have

$$(4.15) \quad \begin{aligned} (a, b, c, c) &= \sigma_1(a, b, c, c) = \tau^3(\sigma_5(a, b, c, c)), \\ (a, c, b, c) &= \sigma_2(a, b, c, c) = \tau^2(\sigma_3(a, b, c, c)), \\ (b, a, c, c) &= \sigma_4(a, b, c, c) = \tau^3(\sigma_6(a, b, c, c)) \end{aligned}$$

and (4.8) becomes

$$(4.16) \quad G_4(a, b, c, c) = G_4(a, c, b, c) = G_4(b, a, c, c) = C_4(a, b, c, c).$$

We use (4.15) and (1.14) to compute (4.9) for $s = 1, 2$ and 3 and divide by $8, 2$ and 2 , respectively. This yields

$$(4.17a) \quad \begin{aligned} G_4(a, b, c, c) + G_4(a, c, b, c) + G_4(b, a, c, c) \\ = 3C_4(a, b, c, c), \end{aligned}$$

$$(4.17b) \quad \begin{aligned} (AB + AC + BC + C^2)G_4(a, b, c, c) \\ + 2(AC + BC)G_4(a, c, b, c) \\ + (AB + AC + BC + C^2)G_4(b, a, c, c) \\ = (2AB + 4AC + 4BC + 2C^2)C_4(a, b, c, c), \end{aligned}$$

$$\begin{aligned}
 (4.17c) \quad & (A^2BC + B^2C^2 + AC^3 + ABC^2)G_4(a, b, c, c) \\
 & + (A^2BC + BC^3 + AB^2C + AC^3)G_4(a, c, b, c) \\
 & + (AB^2C + A^2C^2 + BC^3 + ABC^2)G_4(b, a, c, c) \\
 & = (2A^2BC + 2AB^2C + 2ABC^2 + A^2C^2 + 2AC^3 + B^2C^2 + 2BC^3) \\
 & \cdot C_4(a, b, c, c).
 \end{aligned}$$

Let D be the 3×3 coefficient matrix of the system (4.17a)–(4.17c). The common value $c = d$ must be the maximum, the minimum or the median of the distinct values a, b, c . If it is the maximum, then we may assume without loss of generality that $a < b < c$. Then

$$\begin{aligned}
 (4.18) \quad \det(D) &= \det \begin{vmatrix} 1 & 1 & 1 \\ (AB + \cdots) & (\cdots) & (AB + \cdots) \\ (A^2BC + \cdots) & (A^2BC + \cdots) & (\cdots) \end{vmatrix} \\
 &= A^3B^2C + \text{higher order terms.}
 \end{aligned}$$

If c is the minimum, we may take $a > b > c$. As with (4.18), we obtain

$$(4.19) \quad \det(D) = A^3B^2C + \text{lower order terms.}$$

If c is between a and b , then we may take $a < c < b$. We obtain

$$\begin{aligned}
 (4.20) \quad \det(D) &= \det \begin{vmatrix} 1 & 1 & 1 \\ (AC + \cdots) & (2AC + \cdots) & (AC + \cdots) \\ (\cdots) & (\cdots) & (A^2C^2 + \cdots) \end{vmatrix} \\
 &= A^3C^3 + \text{higher order terms.}
 \end{aligned}$$

Since q is transcendental, we always have

$$(4.21) \quad \det(D) \neq 0$$

and (4.16) follows since the system has a unique solution.

Case (iii). $a = b \neq c = d$.

Observe that

$$\begin{aligned}
 (4.22) \quad & (a, a, c, c) = \sigma_1(a, a, c, c) = \sigma_4(a, a, c, c) \\
 & = \tau^3(\sigma_5(a, a, c, c)) = \tau^3(\sigma_6(a, a, c, c)), \\
 & (a, c, a, c) = \sigma_2(a, a, c, c) = \sigma_3(a, a, c, c).
 \end{aligned}$$

Thus, (4.8) becomes

$$(4.23) \quad G_4(a, a, c, c) = G_4(a, c, a, c) = C_4(a, a, c, c).$$

We use (4.22) and (1.14) to compute (4.9) for $s = 1$ and 2 and divide by 8 and 4, respectively. This gives

$$(4.24a) \quad 2G_4(a, a, c, c) + G_4(a, c, a, c) = 3C_4(a, a, c, c),$$

$$\begin{aligned}
 (4.24b) \quad & (A^2 + 2AC + C^2)G_4(a, a, c, c) + 2ACG_4(a, c, a, c) \\
 & = (A^2 + 4AC + C^2)C_4(a, a, c, c).
 \end{aligned}$$

The determinant of the 2×2 coefficient matrix E of the system (4.24a)–(4.24b) is given by

$$(4.25) \quad \det(E) = \det \begin{vmatrix} 2 & 1 \\ (A^2 + 2AC + C^2) & 2AC \end{vmatrix} = -(A - C)^2,$$

which is not 0 since q is transcendental. Observe that $\det(E)$ can be 0 if q is 0 or a root of unity. (4.23) follows since the system (4.24a)–(4.24b) has a unique solution.

Case (iv). $a = b = c$.

Every permutation of (a, a, a, d) can be obtained by some power of τ . Using (1.14), we simplify (4.9) for $s = 1$ and divide by 24 to obtain

$$(4.26) \quad G_4(a, a, a, d) = C_4(a, a, a, d),$$

as required.

Since some permutation of (a, b, c, d) will fall into one of our four cases, we have established (4.4) and completed our induction for fixed transcendental q . This restriction is easily removed since $G_4(\mathbf{a})$ and $C_4(\mathbf{a})$ are both polynomials in q . \square

5. $n \geq 5$. The q -Dyson conjecture (1.3) is related to the root system A_{n-1} . Since the Weyl group for A_{n-1} is S_n , it is no surprise that (1.3) reduces (see Kadell [25]) to the symmetry (1.9) of the constant term $G_n(\mathbf{a})$. Unfortunately, (1.12) and (4.2) show that the symmetry (1.9) is a difficult problem which may be no easier than (1.3). It seems to recur in a different form in other approaches. In order to use the argument of §4 to treat the case $n \geq 5$, we must find $(n-1)!$ functions $W_n(\mathbf{a})$ satisfying (3.26). It is clear from (3.5) and (3.14a)–(3.17b) that this is not easy. Thus, the complex, asymmetric polynomial $\Sigma_n^*(\mathbf{a}; \mathbf{x})$ acts through (3.26) as a surrogate for the symmetry problem (1.9). Some of the π_α required by Lemma 4 for (3.15a)–(3.16b) must be 4-cycles. This suggests that the q -Dyson conjecture (1.3) involves the cyclic structure of permutations in S_n . That is, it involves the partitions of n . Kadell [25] proves the first part (3.18a) of Conjecture 5 and gives a conjecture which includes (3.18b). If we can find the algebraic secret of the functions $W_n(\mathbf{a})$, then perhaps (3.26) may follow using the techniques of [25].

We must use (1.14) and the equalities among a_i , $1 \leq i \leq n$, to simplify (3.27) and show that some of the resulting equations are linearly independent. Proctor [37] has used linear algebra to solve two difficult combinatorial problems. He does this by showing that certain partially ordered sets are rank-unimodal and have the Sperner property. The second problem is the unimodality of $C_2(\mathbf{a})$ which has often arisen in connection with Lie algebras. See Proctor [37] and Andrews [2, Chapter 3] for extensive references. Another result from partition theory which is related to Lie algebras is the celebrated pair of Rogers-Ramanujan identities. See Lepowsky [27] and Lepowsky and Milne [28].

The q -Dyson conjecture (1.3) also follows from Conjecture 1. A direct cancellation proof of (2.20) is probably as difficult as a sieve proof of the symmetry (1.9). Since Conjecture 1 includes many results, we may try to prove it by induction. Unfortunately, we must add Conjecture 2 to our induction hypothesis, but it is still not enough. Conjectures 1 and 2 imply Conjecture 3. Perhaps we can find a larger family of identities which can be proved by induction. We should at least try to prove Macdonald's [33] result that (1.3) holds when $a_1 = a_2 = \cdots = a_n = 2$ for all $n \geq 2$. We are looking for a multiple series identity which extends the transformation formula of Bailey [6, §4.5(3)]. Gustafson and Milne [19, Theorem 1.7] observe that Good's identity (1.6) is a special case of a theorem which has numerous applications,

particularly to the Holman multiple series introduced by Holman, Biedenharn and Louck [20]. Louck and Biedenharn [29, 30], Biedenharn, Holman and Milne [8] and Milne [35] generalize some results for ordinary hypergeometric functions to Holman series. The q -analogs of these results probably require q -analogs of some expansions involving Schur functions. This may also be the key to an elegant generalization of (1.3) conjectured by Ihrig and Ismail [21] which involves symmetric functions.

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Note added in proof. Zeilberger and Bressoud [Z-B] have recently proved the q -Dyson conjecture. They show that certain ‘bad guys’ contribute 0 to the constant term. Bressoud and Goulden [B-G] modify the proof to treat Conjecture 1 and, consequently, Conjectures 2 and 3.

[Z-B] D. Zeilberger and D. M. Bressoud, *A proof of Andrews’ q -Dyson conjecture* (to appear).

[B-G] D. M. Bressoud and I. P. Goulden, *Constant term identities extending the q -Dyson theorem*, preprint.

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